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A Mirror Pair of Calabi-Yau Fourfolds in Type II String Theory.

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Abstract

We give some simple examples of mirror Calabi-Yau fourfolds in Type II string theory. These are realised as toroidal orbifolds. Motivated by the Strominger, Yau, Zaslow argument we give explicitly the mirror transformation which maps Type IIA/IIB on such a fourfold to Type IIA/IIB on the mirror. The mirrors are related by the inclusion/exclusion of discrete torsion. Implicit in the result is a confirmation of mirror symmetry to genus g in the string path integral. Finally, by considering the relationship between M-theory and Type IIA theory, we show how in M-theory on mirror Calabi-Yau fourfolds, mirror symmetry exchanges the Coulomb branch with (one of the) Higgs branches of the theory. This result is relevant to duality in $N = 2$ supersymmetric gauge theories in three dimensions.

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1 Introduction.

Recent work of Strominger, Yau and Zaslow [1] has provided a deep physical insight into the nature of mirror symmetry for Calabi-Yau 3-folds. This has already had an impact on the mathematical understanding of mirror symmetry [2]. The SYZ argument was generalised in [3] to the case of n-folds. The upshot of the argument is that mirror symmetry is T-duality.

The simplest and perhaps best understood example of a mirror pair of Calabi-Yau 3-folds in string theory was constructed some time ago in [4]. These mirror target spaces are $T^6/Z_2 \times Z_2$ orbifolds and are related by the inclusion or exclusion of discrete torsion. This pair of string target spaces were further analysed in [5], where it was explicitly verified that they are indeed mirrors (This example was discussed again in light of the SYZ argument in [6]). The mirror pair of target spaces considered in [5] are very simple toroidal orbifolds. The purpose of this paper is to consider the SYZ argument applied to string theory with Calabi-Yau fourfold target spaces which are also realised as toroidal orbifolds. This will provide us with a strong confirmation of the SYZ argument.

In fact, all the string target spaces that we consider in this paper occur in the moduli space of the fourfolds constructed (in classical geometry) in [7]. Even though the Hodge numbers of these target spaces have been presented in [7], we will compute them explicitly here using string theory techniques.

The main result of the paper is that we find a strong confirmation of the SYZ argument as well as gaining concrete examples of mirror symmetry for Type II strings on Calabi-Yau fourfolds. The methodology follows that of [5], and the final result is similar in that we obtain an explicit description of the mirror symmetry transformation of the string theory as well as a confirmation of the symmetry to genus g in the string path integral. The key point throughout is that the (T-duality)mirror transformation turns on discrete torsion in the mirror theory, which results in the (mirror) difference between the Hodge numbers of the target spaces. Similar examples to those studied here, but with target spaces of exceptional holonomy were considered in [9].

The organisation of the paper is as follows. In the next section we consider Type II strings on the simplest Calabi-Yau fourfold (orbifold), without introducing any discrete torsion. We compute the Hodge numbers of this target space. Then, using the SYZ argument, we motivate the T-duality transfor-

mation that we propose implements the mirror transformation of this theory. Using this transformation in section four we show (following [5]) that this transformation has the effect of turning on a certain "amount" of discrete torsion. We compute the Hodge numbers of the dual target space thus verifying that the transformation is indeed the mirror transformation. We then consider all the other possibilities of turning on discrete torsion in this class of string target spaces. These six possibilities result in three additional pairs of mirror fourfold target spaces, all of whose Hodge numbers are identical and self-mirror.

Finally, we consider the relation between Type IIA theory on mirror Calabi-Yau fourfolds and M-theory on the same. This leads to a mirror pair of three-dimensional $N = 2$ theories in which the Coulomb branch is exchanged with one of two Higgs branches and vice-versa.

Mirror symmetry in higher dimensions was first studied in [10]. Various aspects of string theory dualities in Calabi-Yau fourfold compactifications have been discussed in [8, 11, 12, 13, 14].

2 Type II Strings on a Calabi-Yau Fourfold, M.

Consider the 8-torus, T^8 with coordinates $x_1, x_2, ...x_8$. The toroidal identifications on the x_i are $x_i{=}x_i + 1$. Define the discrete isometry group, Γ, with generators α, β, γ as follows.

$$\alpha : x_i = -x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8 \tag{1}$$

$$\beta : x_i = x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8 \tag{2}$$

$$\gamma : x_i = -x_1, -x_2, x_3, x_4, -x_5, -x_6, x_7, x_8 \tag{3}$$

We now consider Type II string theory defined on the above orbifold. We wish to compute the Hodge numbers for this orbifold using the standard techniques [15]. The Hodge numbers for this orbifold were presented in [7] and various aspects of string/M/F-theory compactification on this orbifold were described in [8]. However it is instructive for our purposes to compute them explicitly here.

We will first compute the Betti numbers from which we can compute the Hodge numbers.

To do this we need to calculate the Betti numbers in the untwisted sector and then the twisted sector. It is straightforward to compute that in the untwisted sector the Betti numbers[2]are $b_0 = b_8 = 1$, $b_1 = b_8 = b_3 = b_5 = 0$, $b_2 = b_6 = 4$ and $b_4^+ = b_4^- = 11$. Here the plus/minus signs indicate the number of harmonic four forms which are self/anti-self dual under the eight dimensional Hodge star operator. We now turn to the twisted sector.

In general (for abelian orbifolds), there exists a twisted sector for each element of the orbifold isometry group which acts with fixed points. In our example, it is clear that all seven non trivial elements of the group act with fixed points. Of these seven, six invert four coordinates of the torus and one ($\alpha\beta$) inverts all eight coordinates. This means that the $\alpha\beta$ twisted sector is 256-fold degenerate and the other six twisted sectors are 16-fold degenerate. In fact, careful consideration of these latter six twisted sectors shows that they each contribute equally to the Hodge numbers. For this reason we will only consider one of these six and multiply its contribution by a factor of six.

Let us compute one of these six twisted sectors. For definiteness we will consider α. The element α acts on T^8 and fixes 16 four-tori, T^4. These have coordinates x_5, x_6, x_7, x_8. Each T^4 has its Betti numbers given by $(b_0, b_1, b_2, b_3, b_4) = (1, 4, 6, 4, 1)$. However, because α inverts four toroidal coordinates, the shift in zero point energies means that in the full orbifold these b_i contibute to b_{i+2}. This would give the full contribution of α to the Betti numbers, *if* α were the only non-trivial element in the group. However, as this is not the case, we must project onto states which are invariant under the whole group.

To do this we need only consider how all the group elements act on the coordinates of the fixed four-tori (ie $x_5, ...x_8$) and project out the elements of cohomology which are not invariant. It is then straightforward to compute that the Γ invariant harmonic forms of each fixed four-torus contribute $(1, 0, 2, 0, 1)$ to $(b_2, b_3, b_4, b_5, b_6)$. As α fixes sixteen four-tori, its final contribution to the spectrum of Betti numbers is $(16, 0, 32, 0, 16)$. As we noted above, there are five additional twisted sectors which make an identical contribution to the spectrum. Summing these together gives a total contribution of $(96, 0, 192, 0, 96)$ to $(b_2, ...b_6)$, where we have taken into account the shift in zero point energy.

[2]These are simply given by the numbers of independent harmonic forms on T^8 that are invariant under Γ.

The last twisted sector is that associated with $\alpha\beta$. $\alpha\beta$ acts with 256 points on T^8. The cohomology of each of these points consists of a single zero form. However, because $\alpha\beta$ inverts eight coordinates, these appear as four-forms in the cohomology of the orbifold. In fact, these states contribute 256 to b_4^+ (This is essentially because the cohomology of each fixed point is identical and of a definite parity). Adding all these contributions we find that the spectrum of Betti numbers is

$$b_0 = b_8 = 1 \tag{4}$$
$$b_1 = b_7 = 0 \tag{5}$$
$$b_2 = b_6 = 100 \tag{6}$$
$$b_3 = b_5 = 0 \tag{7}$$
$$b_4^+ = 363 \tag{8}$$
$$b_4^- = 107 \tag{9}$$

Let us define complex coordinates (z_i) on the orbifold as follows.

$$z_i = x_i + ix_{i+1} \tag{10}$$

where $i = 1, 3, 5, 7$.

From the definition of the orbifold, it is clear that the holomorphic 4-form is preserved. The target space of the string is therefore a Calabi-Yau fourfold (orbifold). Moreover, using the formula for the Dirac index A on an 8-manifold,

$$24A = -1 + b_1 - b_2 + b_3 + b_4^+ - 2b_4^- \tag{11}$$

we find that A=2; using theorem C of [16], the fourfold has holonomy precisely $SU(4)$.

For a Calabi-Yau fourfold with $SU(4)$ holonomy, we can write the Hodge numbers, $h^{p,q}$ in terms of the Betti numbers as follows.

$$b_0 = h^{0,0} = 1 \tag{12}$$
$$b_2 = h^{1,1} \tag{13}$$
$$b_3 = 2h^{2,1} \tag{14}$$
$$b_4 = b_4^+ + b_4^- = 2(h^{4,0} + h^{3,1}) + h^{2,2} = 2 + 2h^{3,1} + h^{2,2} \tag{15}$$

We can also use the fact that the number of real metric moduli associated to metrics of $SU(4)$ holonomy is

$$b_4^- + 1 = h^{1,1} + 2h^{3,1} \tag{16}$$

These are the numbers of Kahler and complex structure deformations which preserve the Calabi-Yau conditions.

From these relations, and our computation of the Betti numbers it follows that

$$h^{1,1} = 100 \tag{17}$$
$$h^{2,1} = 0 \tag{18}$$
$$h^{3,1} = 4 \tag{19}$$
$$h^{2,2} = 460 \tag{20}$$
$$h^{0,0} = h^{4,0} = 1 \tag{21}$$

This is in agreement with [7].

3 The (T-duality) Mirror Transformation.

In the next section, we will T-dualise the orbifold of the last section. This transformation will coincide with the mirror transformation. The T-duality transformation we will use will be motivated by the Strominger, Yau, Zaslow argument [1] concerning the implications of quantum mirror symmetry for Calabi-Yau threefolds, generalised to the case of fourfolds [3]. The SYZ argument implies that any Calabi-Yau n-fold for which a mirror exists admits a supersymmetric T^n fibration. Mirror symmetry is then just T-duality on the T^n fibers. For us, this means that if we consider Type IIA or IIB string theory defined on M, and assume the existence of a mirror target space (M'), $R \to 1/R$ T-duality on the T^4 fibers, will map us back to the IIA or IIB theory (respectively) on M'.

Since M is defined as the target space of the string theory on an orbifold, the T-duality/mirror transformation must be definable on the classical orbifold T^8/Γ. This then leaves us with the problem of identifying which T^4 submanifold of T^8/Γ to T-dualise. To solve this problem, we need the definition of the supersymmetric four-torus itself.

In supersymmetric compactification of any superstring theory or M-theory on some manifold S, the supersymmetric cycles are identified with (homology) volume minimising cycles. This is because if we consider branes wrapping some cycle, D in S, the mass of the state which appears in the non-compact directions is proportional to the volume of D. If the cycle has minimal volume, we get a state of minimum mass and hence the states corresponding to minimal volume cycles are (supersymmetric) BPS states. The mathematical theory of minimal submanifolds was explored in [17]. There it was shown that if S admits some globally defined form (K), and K restricted to D is proportional to the volume form of D, then D is volume minimising within its homology class[3]. D is then said to be callibrated by K. The conditions for supersymmetry for branes wrapped around cycles have been analysed in [18, 19, 6, 3]. There it has been shown that (for compactification manifolds which preserve supersymmetry), the supersymmetric cycles are precisely the callibrated submanifolds in [17].

For Calabi-Yau n-folds with mirrors, the supersymmetric n-tori which fiber the whole manifold are callibrated by $Re(\omega)$, where ω is the holomorphic n-form. In the case we are considering, ω is of degree four, so the supersymmetric cycles callibrated by $Re(\omega)$ are four-cycles. The four-cycle we are interested in must be a four-torus as follows from the SYZ argument [1]. Moreover, because we have the freedom to fix the moduli in the twisted sector, this four-torus should exist in T^8/Γ viewed as an orbifold in classical geometry. The four-torus is therefore a four-cycle in T^8 which is preserved by Γ. As we have noted in the previous section, there are 22 invariant four-cycles, and one of these is the one which interests us. Since we will be considering T-duality on T^8/Γ, viewed as a complex orbifold with complex coordinates z_i, we can appeal to what we understand about T-duality and mirror symmetry for complex tori[4].

For a complex torus with coordinates z_i, mirror symmetry just corresponds to $R \to 1/R$ T-duality on the circles with coordinates $Im(z_i)$. It is natural to expect that precisely this transformation applies to T^8/Γ. This was the case for the Calabi-Yau threefold orbifold considered in [5]. We are therefore lead to consider the four-torus in T^8/Γ with real coordinates x_2, x_4, x_6, x_8. It is gratifying to see that $Re(\omega)$ restricted to this cycle is the

[3]Clearly, for this to be so, the degree of K must be the dimension of D.

[4]This was reviewed in [5].

volume form. It is clear then how to proceed. We will consider Type IIA (or IIB) string theory defined on the orbifold M of the previous section and consider $R \rightarrow 1/R$ T-duality transformations in the $2,4,6,8$ directions. This should map us to Type IIA (or IIB) theory on a target space with Hodge numbers mirror to those of M. We will now verify that this is the case.

4 Type II Strings on the Mirror of M.

We have been lead to consider T-dualising the circles in T^8/Γ with coordinates (x_2, x_4, x_6, x_8). Let us refer to this transformation as T. As was pointed out in [5], T-duality for orbifolds such as ours can result in the turning on of discrete torsion [20] in the T-dual orbifold. In [5], it was discrete torsion that was responsible for "producing" the mirror of a Calabi-Yau threefold. This will be the case here. However, in the case considered in [5], the orbifold isometry group was $Z_2 \times Z_2$ and hence the discrete torsion was Z_2 valued[5]. Thus, in [5], the discrete torsion was unique. For our case, the discrete torsion is not unique and hence there exist several possibilities. In fact there exist seven non-trivial modular invariant possibilities for what the discrete torsion may be. However, we will see that under the transformation T, just the right "amount" of discrete torsion will be present in the T-dual orbifold to produce the mirror of M. We will now derive the discrete torsion phase factor which appears in the path integral measure when we apply the transformation T to the string theory on M. The discussion follows [5] which we will briefly review here.

Consider the n-torus with coordinates $(x_1, x_2, ...x_n)$. Superstrings propagating on the torus have coordinates X_i with fermionic partners ψ^i. Under an $R \rightarrow 1/R$ T-duality transformation in any one of the n-directions, the measure of the genus g path integral for the theory, μ_g transforms as follows.

$$\mu_g \longrightarrow (-1)^{\sigma_\alpha} \mu_g \tag{22}$$

where

$$\alpha = (\theta_i, \phi_j) \tag{23}$$

[5]The group in which the discrete torsion takes values corresponds to $H^2(G, U(1))$ [20], where G is the orbifold group.

is the spin structure of the genus g Riemman surface (relative to the canonical (a_i, b_j) basis of 1-cycles) and

$$\sigma = \theta.\phi \tag{24}$$

is its parity. We can write θ_i=Θ, a g-vector and similarly for ϕ. If we T-dualise k directions, the factor of $(-1)^\sigma$ appears k times. These factors are responsible for the T-duality of Type IIA/IIB string theories compactified on n-tori.

In the orbifold that we are discussing the above transformation rule gets modified due to the twist by Γ in the following way.

With the generators of Γ as we have defined them, a general twist of the genus g Riemman surface in the (a_i, b_j) directions involves a pair of elements that we can write as $(\alpha^R\beta^S\gamma^T, \alpha^U\beta^V\gamma^W)$. Here, R, S, T, U, V, W are g-vectors which describe the change in boundary conditions of the ψ_i when going around the 1-cycles on the genus g world sheet. Under a T-duality transformation in the rth direction the measure will pick up a factor corresponding to a shifted spin structure. For example, if we begin with Type IIA theory on our original orbifold and T-dualise the circle with coordinate x_8, the spin structure gets shifted as follows:

$$(\Theta, \Phi) \longrightarrow (\Theta + S, \Phi + V) \tag{25}$$

This is because β is the only generator which acts on x_8 and its action is just a minus sign. The genus g measure therefore picks up a factor which is (-1) raised to the parity of this shifted spin structure.

In the problem we are addressing, we are dualising the $(2, 4, 6, 8)$ directions and all we need to do now is calculate all the factors which appear in the T transformed path integral. These are the parities of the shifted spin structures in the dualised directions. These are:

$$x_2 : \sigma(\Theta + R + T, \Phi + U + W) \tag{26}$$

$$x_4 : \sigma(\Theta + R, \Phi + U) \tag{27}$$

$$x_6 : \sigma(\Theta + S + T, \Phi + V + W) \tag{28}$$

$$x_8 : \sigma(\Theta + S, \Phi + V) \tag{29}$$

Each of these factors appears as a power of (-1) in the T transformed path integral measure. A useful formula for shifted spin structures is [21]:

$$\sigma(\Theta+A+B, \Phi+C+D) = \sigma(\Theta, \Phi)+\sigma(\Theta+A, \Phi+C)+\sigma(\Theta+B, \Phi+D)+A.D-B.C \tag{30}$$

Using this we find that the path integral measure transforms under T as

$$\mu_g \longrightarrow (-1)^{R.W-T.U+S.W-T.V}.\mu_g \tag{31}$$

This formula shows that under T, the IIA/IIB theory on the orbifold with generators Γ transforms into the IIA/IIB theory with generators Γ but with discrete torsion (given by the above factor) turned on. We will call the T-dual target space M'.

4.1 The Hodge Numbers of M'.

In this subsection we will calculate the Hodge numbers for the string theory on M'. Discrete torsion has no effect on the untwisted sector of an orbifold. The contribution from this sector is therefore the same as in section 2.

Let us consider the α twisted sector. As before, α fixes sixteen 4-tori. We then have to project on to states invariant under Γ. This is where the effect of discrete torsion comes into being. In general, certain states which would have survived the Γ projection in the absence of discrete torsion will now remain invariant (and vice-versa). This is most easily seen in the Hamiltonian framework [5]. Let us consider the σ and τ directions of the world sheet as space and time. As we have noted, the general twist of the world sheet in these directions is given by $(\alpha^R\beta^S\gamma^T, \alpha^U\beta^V\gamma^W)$. For a fixed τ, we have a Hilbert space of states consisting of all the possible twisted states associated with Γ. However, as τ evolves, because of the projection by Γ in these directions also, we project onto Γ invariant states. For example, the element (α, β) corresponds to a sector of the Hilbert space consisting of the α twisted sector projected onto β invariant states. If, for this element the discrete torsion factor is (-1), then we include an extra minus sign in the action of β on the α twisted sector, relative to its "geometric" action in the absence of the torsion.

Let us consider the α twisted sector. This corresponds to elements of Γ of the form $(\alpha, \alpha^U\beta^V\gamma^W)$. Our discrete torsion in a general sector is of the form

$$\epsilon = (-1)^{R.W-T.U+S.W-T.V} \tag{32}$$

Thus, in the α twisted sector the discrete torsion factor is

$$\epsilon = (-1)^W \tag{33}$$

From this it follows that we have to insert an extra minus sign in the action of elements of Γ which contain an odd number of γ's when acting on the cohomology of the sixteen 4-tori fixed by α. These elements are $(\gamma, \alpha\gamma, \beta\gamma, \alpha\beta\gamma)$. The remaining four elements act with their standard geometric action as if the discrete torsion was not present. Putting these facts together, it is straightforward to compute that in the orbifold with discrete torsion, the contribution of each 4-torus fixed by α to the Betti numbers is to add $(0, 0, 4, 0, 0)$ to $(b_2, b_3, b_4, b_5, b_6)$[6]. Moreover, of these four-forms, two are self-dual/anti self-dual. This gives a total contribution to the Betti numbers in the α twisted sector of 32 self-dual and 32 anti self-dual four forms.

Considering carefully the other five twisted sectors which are sixteen-fold degenerate, we find that these contribute an identical spectrum to that of α. However, we would like to stress that this is a highly non-trivial fact and is not dictated by any obvious symmetry arguments and needs to be checked explicitly.

Finally, we need to consider the $\alpha\beta$ twisted sector. This corresponds to elements of the form $(\alpha\beta, \alpha^U \beta^V \gamma^W)$, for which it is straightforward to see that the discrete torsion is trivial. This sector therefore makes the same contribution to the spectrum as we found in section 2, which is 256 to b_4^+.

Combining all these contributions we find that the relevant Betti numbers of M' are:

$$(b_2, b_3, b_4^+, b_4^-) = (4, 0, 459, 203) \tag{34}$$

Using the formulas of section 2, we find that $A = 2$ and

$$(h^{1,1}, h^{2,1}, h^{3,1}, h^{2,2}) = (4, 0, 100, 460) \tag{35}$$

We can therefore see that $h^{p,q}(M) = h^{4-p,q}(M')$. Hence the Hodge numbers of the two target spaces are mirror to one another. We conclude that T is indeed the mirror symmetry transformation.

4.2 Further Examples.

As we pointed out in the previous subsection, there exist eight a priori different orbifolds with generators (α, β, γ) given in equations (1) – (3). This

[6] If we denote by x_{ij} the two-form, $dx_i \wedge dx_j$ on the fixed T^4, then the two-forms x_{ij} for $(ij) = [(57), (58), (67), (68)]$ are invariant. Because of the zero point energy in the twisted sector, these contribute to the spectrum of harmonic four-forms on M'.

is due the fact that in addition to the orbifold without discrete torsion, we can turn on seven non-trivial discrete torsion factors consistent with modular invariance. In fact the discrete torsion group in this case is $Z_2 \times Z_2 \times Z_2$, generated by $[(-1)^{R.V-U.S}, (-1)^{R.W-T.U}, (-1)^{S.W-T.V}]$. It is natural to consider the mirror symmetry transformation T in all these cases. Let us define the three generators of the discrete torsion group as (g_1, g_2, g_3) respectively. Let us also denote the target space of the orbifold theory with discrete torsion factor $k \equiv ({g_1}^a.{g_2}^b.{g_3}^c)$ by M_k. In this notation, the orbifold without discrete torsion and its mirror are denoted by M_1 and $M_{g_2.g_3}$ respectively.

It is clear from the discussion of the previous subsection that under T, any of the eight target spaces M_k will transform under T to another of the M_k. In fact T produces four pairs of target spaces, which in principle are mirrors of one another, ie the eight M_k "transform" as four doublets under T. It is also clear that, under T, the target spaces M_k transform as follows:

$$M_k \longrightarrow M_{k.g_2.g_3} \tag{36}$$

The four pairs of target spaces related by T are therefore given by pairs $(k, k.g_2.g_3)$. With the expectation that these pairs of target spaces should be mirror to one another we can compute the Hodge numbers for each pair to verify this. This gives the result that apart from the mirror pair (with $k = 1$) discussed above, the Hodge numbers for all other pairs ($k \neq 1$) are identical and self-mirror. In fact the Hodge numbers of M_k, for $k \neq 1, g_2.g_3$ are:

$$(h^{1,1}, h^{2,1}, h^{3,1}, h^{2,2}) = (20, 64, 20, 76) \tag{37}$$

Since these are self-mirror, we get no contradiction with the SYZ argument that lead us to conclude that T is the mirror transformation. We would also like to note that for reasons explained in [5], implicit in the above result is the fact that mirror symmetry between the dual pairs of theories is confirmed to genus g in the string path integral.

5 Relation to M-theory.

In this section we briefly discuss the implications of mirror symmetry for Calabi-Yau fourfolds for $N = 2$ vacua of M-theory in three dimensions.

M-theory compactified on a manifold of $SU(4)$ holonomy gives a three dimensional theory with $N = 2$ supersymmetry. General aspects of such compactifications were studied in [13], whereas the relationship of such vacua (via F-theory) to $N = 1$ vacua in four dimensions has been studied in [11, 12].

Type IIA theory compactified on a Calabi-Yau fourfold (M) has a strong coupling limit (keeping the Calabi-Yau moduli fixed) which is the three dimensional theory obtained from M-theory on M. Moreover if we now consider IIA theory on M', the mirror of M, then it has a limit[7] which is M-theory on M'. This chain of dualities provides a duality between M-theory on M and M'. Let us discuss what this implies for the low energy spectrum.

In uncompactified M-theory the low energy dynamics is described by eleven dimensional supergravity. The relevant degrees of freedom are the 3-form potential A_3 and the metric G. Let us consider the low energy spectrum of M-theory compactified on M and its mirror. We will denote by $h_{p,q}$ the Hodge numbers of M and by $h'_{p,q}$ those of M'. As the two manifolds are mirror, $h_{p,q} = h'_{4-p,q}$.

In M-theory on M, the three form contributes $h_{1,1}$ massless vector fields and $2h_{2,1}$ scalars. The metric contributes $h_{1,1}$ scalars (Kahler moduli) and $2h_{3,1}$ further scalars (complex structure moduli). All these massless fields combine into the following multiplets of $N = 2$ supersymmetry. The $h_{1,1}$ vectors and scalars combine to give $h_{1,1}$ vector multiplets, V_i. The scalars from A_3 form $h_{2,1}$ scalar multiplets, S_s. The complex structure moduli from the metric form $h_{3,1}$ additional scalar multiplets, H_h. Vectors in three dimensions are dual to scalars so we can dualise the vectors to form $h_{1,1}$ scalar multiplets which we also denote by V_i. The V_i parametrise a $2h_{1,1}$ dimensional Coulomb branch, V. The other scalars parametrise two Higgs branches, H_1 and H_2 of respective dimensions $2h_{3,1}$ and $2h_{2,1}$.

In M-theory on M' we have a $2h_{3,1}$ dimensional Coulomb branch, V', a $2h_{1,1}$ dimensional Higgs branch ${H_1}'$ and an additional $2h_{2,1}$ dimensional Higgs branch, ${H_2}'$. Mirror symmetry of the Type IIA theory on M and M' implies a mirror symmetry between M-theory on M and M-theory on M'. By a simple counting of the moduli space dimensions of the various branches it is straightforward to see that mirror symmetry is the following exchange

[7] In [22] it was argued that M-theory is devoid of exotic phase transitions to compactifications which are defined in the corresponding IIA theory as abstract conformal field theories with no definite analogue in classical geometry. We will assume in this section that the mirror target spaces are well defined in classical geometry.

symmetry:

$$V \leftrightarrow {H_1}' \tag{38}$$
$$H_1 \leftrightarrow V' \tag{39}$$
$$H_2 \leftrightarrow {H_2}' \tag{40}$$
$$\tag{41}$$

This result has obvious implications for duality in $N = 2$ gauge theories in three dimensions. Examples of such gauge theories have recently been discussed in [23]. It would be interesting to explore this further.

6 Discussion.

In the examples of mirror symmetry that we have discussed in this paper, it is clear that the Strominger-Yau-Zaslow argument [1] (or its generalisation to n-folds [3]) is a very powerful one. In turn we hope that the mirror pairs of string theories constructed here, being so simple, can be used to check other properties of mirror symmetry in higher dimensions [10], and may even provide insights into other properties of higher dimensional mirror symmetry.

The relationship between mirror symmetry for fourfolds and $N = 2$ M-theory compactifications is also very promising for future investigations.

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